18. Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli’s Law states that the rate of change of volume, $V$, of water in the tank is proportional to the square root of the height, $h$, of the water above the hole.

This is given by the differential equation:

\[ \frac{dV}{dt} = -k\sqrt{h}, \quad k > 0. \]

(a) For a cylindrical tank with constant cross-sectional area, $A$, show that the rate of change of the height of the water in the tank is given by

\[ \frac{dh}{dt} = \frac{-k}{A}\sqrt{h}. \]

(b) Initially, when the height of the water is 144 cm, the rate at which the height is changing is $-0.3$ cm/hr.

By solving the differential equation in part (a), show that $h = \left(12 - \frac{1}{80}t\right)^2$.

(c) How many days will it take for the tank to empty?

(d) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in cm$^3$/hr) at the end of the fourth day.
Answers

(a) **Method 1**

\[ V = Ah \text{ (here of below)} \]

\[ \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} \]

\[ \frac{dV}{dh} = A \]

\[ \therefore \frac{dh}{dV} = \frac{1}{A} \]

\[ = \frac{1}{A} \cdot -k \sqrt{h} \]

\[ = \frac{-k}{A} \sqrt{h} \]

**Method 2**

\[ V = Ah \]

\[ \frac{dV}{dt} = \frac{d}{dt} (Ah) \]

\[ \frac{dV}{dh} = A \]

\[ \therefore \frac{dh}{dV} = \frac{1}{A} \]

\[ = \frac{1}{A} \cdot -k \sqrt{h} \]

\[ = \frac{-k}{A} \sqrt{h} \]

**Method 3**

\[ \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \]

\[ -k \sqrt{h} = A \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{-k}{A} \sqrt{h} \]

(b) \[ \frac{dh}{dt} = -0.3 \text{ cm/hr when } h = 144 \]

\[ -0.3 = \frac{k}{A} \sqrt{144} \]

\[ \frac{k}{A} = \frac{1}{40} \therefore A = 40k \]

\[ \frac{dh}{dt} = \frac{-k}{A} \sqrt{h} \]

\[ \int \frac{1}{\sqrt{h}} \, dh = \int \frac{-k}{A} \, dt \]

\[ 2\sqrt{h} = \frac{-k}{A} t + c \]

\[ 2\sqrt{144} = c \quad c = 24 \]

\[ 2\sqrt{h} = \frac{-k}{A} t + 24 \]

\[ \sqrt{h} = \frac{-k}{2A} t + 12 \]

\[ h = \left(\frac{-k}{2A} t + 12\right)^2 \]

\[ h = \left(\frac{-1}{80} t + 12\right)^2 \]

(c) \[ 0 = \left(\frac{-1}{80} t + 12\right)^2 \]

\[ \frac{-1}{80} t + 12 = 0 \]

\[ t = 960 \text{ hours} \]

\[ \text{number days} = \frac{960}{24} = 40 \text{ days} \]

(d) \[ A = 400 \pi \]

\[ \frac{k}{A} = \frac{1}{40} \]

\[ k = 10 \pi \]

\[ h = \left(\frac{-1}{80} \cdot 96 + 12\right)^2 \]

\[ \frac{dV}{dt} = -108 \pi \]

\[ \therefore \text{ Rate to vegetation is } 108\pi \text{ cm}^3/\text{hr} \]