2015 Q14

14. For some function, \( f \), define

\[
g(x) = f(x) + f(-x) \quad \text{and} \quad h(x) = f(x) - f(-x).
\]

Show that \( g(x) \) is an even function and that \( h(x) \) is an odd function.
Hence show that \( f(x) \) can be expressed as the sum of an even and an odd function.

Answers

\[
g(x) = f(x) + f(-x)
\]
\[
g(-x) = f(-x) + f(x) = f(x) + f(-x) = g(x)
\]

\[
.: \text{since } g(-x) = g(x) \text{ function is even.}
\]
\[
h(x) = f(x) - f(-x)
\]
\[
h(-x) = f(-x) - f(x) = -f(x) + f(-x) = -[f(x) - f(-x)] = -h(x)
\]

\[
.: \text{since } h(-x) = -h(x) \text{ function is odd.}
\]
\[
g(x) + h(x) = 2f(x) \text{ by adding initial equations}
\]
\[
f(x) = \frac{1}{2}g(x) + \frac{1}{2}h(x)
\]

\[
.: \text{Since } g \text{ even and } h \text{ odd, } f(x) \text{ is the sum of an even and an odd functions.}
\]