16.  
(a) Express $-1$ as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$.

(b) Write down the four solutions to the equation $z^4 - 1 = 0$.

(c) Plot the solutions of both equations on an Argand diagram.

(d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$.

(e) Hence identify all the solutions to the equation $z^6 + z^4 + z^2 + 1 = 0$.

Answers

(a) $z = \cos\left(\frac{\pi}{4}\right) \pm i \sin\left(\frac{\pi}{4}\right), \quad \cos\left(\frac{3\pi}{4}\right) \pm i \sin\left(\frac{3\pi}{4}\right)$

(b) $z = \pm i, \quad \pm 1$

(c)

(d) $z^8 - 1 = (z^4 + 1)(z^4 - 1)$

Then the solutions to $z^4 + 1 = 0$ and $z^4 - 1 = 0$ are also the solutions to $z^8 - 1 = 0$.

(e) Observe that $z^6 + z^4 + z^2 + 1 = (z^2 + 1)(z^4 + 1)$

OR

$z^8 - 1 = (z^4 + 1)(z^2 + 1)(z^2 - 1)$

∴ Six solutions are those above except $z = \pm 1$