15. Lines $L_1$ and $L_2$ are given by the parametric equations

$$L_1 : x = 2 + s, \ y = -s, \ z = 2 - s \quad L_2 : x = -1 - 2t, \ y = t, \ z = 2 + 3t.$$ 

(a) Show that $L_1$ and $L_2$ do not intersect.

(b) The line $L_3$ passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both $L_1$ and $L_2$. Obtain parametric equations for $L_3$.

(c) Find the coordinates of the point $Q$ where $L_3$ and $L_2$ intersect and verify that $P$ lies on $L_1$.

(d) $PQ$ is the shortest distance between the lines $L_1$ and $L_2$. Calculate $PQ$.

Answers

See over the page
Answers

(a) Equating the \( x \)-coordinates: \( 2 + s = -1 - 2t \) \( \Rightarrow \ s + 2t = -3 \) (1)
Equating the \( y \)-coordinates: \( -s = t \) \( \Rightarrow \ s = -t \)
Substituting in (1): \( -t + 2t = -3 \) \( \Rightarrow \ t = -3 \) \( \Rightarrow \ s = 3 \).
Putting \( s = 3 \) in \( L_1 \) gives \( (5, -3, -1) \) and \( t = -3 \) in \( L_2 \), \( (5, -3, -7) \).
As the \( z \) coordinates differ, \( L_1 \) and \( L_2 \) do not intersect.

(b) Directions of \( L_1 \) and \( L_2 \) are: \( \mathbf{i} - \mathbf{j} - \mathbf{k} \) and \( -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \). The vector product of these gives the direction of \( L_3 \).

\[
(i - j - k) \times (-2i + j + 3k) = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2i - j - k
\]
Equation of \( L_3 \):

\[
\mathbf{r} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} + (-2\mathbf{i} - \mathbf{j} - \mathbf{k})u = (1 - 2u)\mathbf{i} + (1 - u)\mathbf{j} + (3 - u)\mathbf{k}
\]
Hence \( L_3 \) is given by \( x = 1 - 2u, y = 1 - u, z = 3 - u \).

(c) Solving the \( x \) and \( y \) coordinates of \( L_3 \) and \( L_2 \):

\[-1 - 2t = 1 - 2u \text{ and } t = 1 - u \Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0 \]
The point of intersection, \( Q \), is \((-1, 0, 2)\) since \( 2 + 3t = 2 \) and \( 3 - u = 2 \).
\( L_1 \) is \( x = 2 + s, y = -s, z = 2 - s \). When \( x = 1, s = -1 \) and hence \( y = 1 \) and \( z = 3 \), i.e. \( P \) lies on \( L_1 \).

(d) \( PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6} \).