### Vectors

**AH Maths Exam Questions**

**Source:** 2019 Specimen P2 Q13 AH Maths

#### (1)

A line, \( L \), has equation \( \frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1} \).

(a) Find the Cartesian equation of the plane, perpendicular to the line \( L \), which passes through the point \( P(1,1,0) \).

(b) Find the shortest distance from \( P \) to \( L \) and explain why this is the shortest distance.

#### Answers:

<p>| | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>find normal vector</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>substitute into equation of the plane</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>find the equation of plane</td>
<td>3</td>
</tr>
<tr>
<td>(b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>find parametric equations for the line</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>substitute into equation of plane</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>solve for ( t )</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>calculate coordinates</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>components of ( PQ )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>find shortest distance</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>explanation</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
</tbody>
</table>
The equations of two planes are given below.

\[ \pi_1: \ 2x - 3y - z = 9 \]
\[ \pi_2: \ x + y - 3z = 2 \]

(a) Verify that the line of intersection, \( L_1 \), of these two planes has parametric equations

\[ x = 2\lambda + 3 \]
\[ y = \lambda - 1 \]
\[ z = \lambda \]

(b) Let \( \pi_3 \) be the plane with equation \(-2x + 4y + 3z = 4\).
Calculate the acute angle between the line \( L_1 \) and the plane \( \pi_3 \).

(c) \( L_2 \) is the line perpendicular to \( \pi_3 \) passing through \( P(1, 3, -2) \).
Determine whether or not \( L_1 \) and \( L_2 \) intersect.

### Answers:

<table>
<thead>
<tr>
<th>(a)</th>
<th>1 verify that the line lies on one plane</th>
<th>1 ( \frac{1}{2} ) ( 2 \lambda + 3 \rightarrow 3(\lambda - 1) - \lambda = 9 ) &lt;br&gt;2 verify for other plane and state conclusion</th>
<th>1 ( \frac{1}{2} ) ( 2 \lambda + 3 + \lambda - 1 - 3\lambda = 2 ); therefore the line lies on both planes &lt;br&gt;( \frac{1}{2} ) ( 2 \lambda + 3 \rightarrow 3(\lambda - 1) - \lambda = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>3 identify vectors</td>
<td>3 ( \left( \begin{array}{c} 2 \ 1 \ 1 \end{array} \right) \rightarrow \left( \begin{array}{c} -2 \ 4 \ 3 \end{array} \right) )</td>
<td>4 ( \cos \theta = \left( \frac{3}{\sqrt{6/929}} \right) )</td>
</tr>
<tr>
<td></td>
<td>4 start to calculate angle</td>
<td>4 ( \cos \theta = \left( \frac{3}{\sqrt{6/929}} \right) )</td>
<td>5 any answer which rounds to 0.229 or ( 13^\circ )</td>
</tr>
<tr>
<td></td>
<td>5 calculate complement</td>
<td>5 any answer which rounds to 0.229 or ( 13^\circ )</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>6 parametric equations for ( L_2 )</td>
<td>( x = -2\mu + 1; \ y = 4\mu + 3; \ z = 3\mu - 2 )</td>
<td>7 any two from ( 2\lambda + 3 = -2\mu + 1; \ \lambda - 1 = 4\mu + 3; \ \lambda = 3\mu - 2 )</td>
</tr>
<tr>
<td></td>
<td>7 two equations for two parameters</td>
<td>7 any two from ( 2\lambda + 3 = -2\mu + 1; \ \lambda - 1 = 4\mu + 3; \ \lambda = 3\mu - 2 )</td>
<td>8 ( \mu = -1; \ \lambda = 0 )</td>
</tr>
<tr>
<td></td>
<td>8 solve for two possible parameters</td>
<td>8 ( \mu = -1; \ \lambda = 0 )</td>
<td>9 ( \frac{1}{2} ) ( \text{eg LHS} = 0, \text{RHS} = -5 ) \text{so lines do not intersect.}</td>
</tr>
<tr>
<td></td>
<td>9 substitute into remaining equation and state conclusion</td>
<td>9 ( \frac{1}{2} ) ( \text{eg LHS} = 0, \text{RHS} = -5 ) \text{so lines do not intersect.}</td>
<td></td>
</tr>
</tbody>
</table>
(3) Planes \( \pi_1, \pi_2 \) and \( \pi_3 \) have equations:

\[
\begin{align*}
\pi_1 & : x - 2y + z = -4 \\
\pi_2 & : 3x - 5y - 2z = 1 \\
\pi_3 & : -7x + 11y + az = -11
\end{align*}
\]

where \( a \in \mathbb{R} \).

(a) Use Gaussian elimination to find the value of \( a \) such that the intersection of the planes \( \pi_1, \pi_2 \) and \( \pi_3 \) is a line.

(b) Find the equation of the line of intersection of the planes when \( a \) takes this value.

The plane \( \pi_4 \) has equation \(-9x + 15y + 6z = 20\).

(c) Find the acute angle between \( \pi_1 \) and \( \pi_4 \).

(d) Describe the geometrical relationship between \( \pi_2 \) and \( \pi_4 \).

Justify your answer.

**Answers:**

| (a) | \( \bullet^1 \) set up augmented matrix | \( \bullet^1 \) \[
\begin{bmatrix}
1 & -2 & 1 & -4 \\
3 & -5 & -2 & 1 \\
-7 & 11 & a & -11
\end{bmatrix}
\]
| \( \bullet^2 \) obtain two zeros | \( \bullet^2 \) \[
\begin{bmatrix}
1 & -2 & 1 & -4 \\
0 & 1 & -5 & 13 \\
0 & 0 & a + 7 & -39
\end{bmatrix}
\]
| \( \bullet^3 \) complete row operations | \( \bullet^3 \) \[
\begin{bmatrix}
1 & -2 & 1 & -4 \\
0 & 1 & -5 & 13 \\
0 & 0 & a - 8 & 0
\end{bmatrix}
\]
| \( \bullet^4 \) obtain value for \( a \) | \( \bullet^4 \) \( a = 8 \) |
(b) 

\[5. \text{ introduce parameter and substitute} \]

\[6. \text{ equation of line} \]

\[5. \quad z = t, \quad y - 5t = 13 \]

\[6. \quad x = 22 + 9t, \quad y = 13 + 5t, \quad z = t \]

(c) 

\[7. \text{ write down normals} \]

\[8. \text{ start to find angle} \]

\[9. \text{ find acute angle} \]

\[7. \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \text{ stated or implied} \]

\[8. \quad \cos \theta = \frac{-11}{\sqrt{38} \sqrt{6}} \text{ OR } \cos \theta = \frac{11}{\sqrt{38} \sqrt{6}} \]

\[9. \quad 0.75 \]

(d) 

\[10. \text{ explanation} \]

\[10. \quad \text{ Planes } \pi_2 \text{ and } \pi_4 \text{ are parallel because the normal of } \pi_4 \text{ is a multiple of the normal of } \pi_2. \]
(a) A beam of light passes through the points B(7, 8, 1) and T(−3, −22, 6).

Obtain parametric equations of the line representing the beam of light.

(b) A sheet of metal is represented by a plane containing the points P(2, 1, 9),
Q(1, 2, 7) and R(−3, 7, 1).

Find the Cartesian equation of the plane.

(c) The beam of light passes through a hole in the metal at point H.

Find the coordinates of H.

---

Answers:

(a)

<table>
<thead>
<tr>
<th>1. Obtain direction vector 1,2,4</th>
<th>[ \mathbf{d} = \begin{pmatrix} 2 \ 6 \ -1 \end{pmatrix} ] or multiple thereof</th>
</tr>
</thead>
</table>
| 2. State parametric equations 3,4,5 | \[ x = 2\lambda + 7 \]
\[ y = 6\lambda + 8 \]
\[ z = -\lambda + 1 \]

or

\[ x = 2\lambda - 3 \]
\[ y = 6\lambda - 22 \]
\[ z = -\lambda + 6 \]

Or equivalent
(b)

3. identify vectors

3. any two from \( \overrightarrow{PQ} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} -5 \\ 6 \\ -8 \end{pmatrix} \),

\( \overrightarrow{QR} = \begin{pmatrix} -4 \\ 5 \\ -6 \end{pmatrix} \) or equivalent

4. evidence of strategy for finding normal

4. \( \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ -5 & 6 & -8 \end{vmatrix} \) or equivalent

5. calculate normal

5. \( \mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \)

6. obtain equation

6. \( 4x + 2y - z = 1 \)

(c)

7. substitute into equation of plane

7. \( 4(2\lambda + 7) + 2(6\lambda + 8) - (-\lambda + 1) = 1 \)

8. find \( \lambda \)

8. \( \lambda = -2 \)

9. determine coordinates of H

9. \( H(3, -4, 3) \)
Two lines $L_1$ and $L_2$ are given by the equations:

$L_1$: \[ x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda \]

$L_2$: \[ \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3} \]

(a) Show that the lines $L_1$ and $L_2$ intersect and find the point of intersection.

(b) Calculate the obtuse angle between the lines $L_1$ and $L_2$.

Answers:

<table>
<thead>
<tr>
<th>(a)</th>
<th>1 convert any two components of $L_2$ to parametric form 1</th>
<th>1 two from $x = 3 - 2\mu$, $y = 8 + \mu$, $z = -1 + 3\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 two linear equations involving two distinct parameters</td>
<td>2 two from $4 + 3\lambda = 3 - 2\mu$, $2 + 4\lambda = 8 + \mu$, $-7\lambda = -1 + 3\mu$</td>
</tr>
<tr>
<td></td>
<td>3 find parameter values</td>
<td>3 $\lambda = 1$, $\mu = -2$</td>
</tr>
<tr>
<td></td>
<td>4 verify third component in both equations or equivalent</td>
<td>4 eg $z_1 = -7 \times 1$ and $z_2 = 3(-2) - 1$ therefore the lines intersect</td>
</tr>
<tr>
<td></td>
<td>5 find point of intersection</td>
<td>5 $(7, 6, -7)$</td>
</tr>
</tbody>
</table>
\( \mathbf{d}_1 = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} \)

\( \mathbf{d}_2 = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \)

\( |\mathbf{d}_1| = \sqrt{74}, \quad |\mathbf{d}_2| = \sqrt{14} \) and

\( \mathbf{d}_1 \cdot \mathbf{d}_2 = -6 + 4 - 21 = -23 \)

\( \cos^{-1} \left( \frac{-23}{\sqrt{74}\sqrt{14}} \right) \approx 135.6^\circ \)
A line, \( L_1 \), passes through the point \( P(2, 4, 1) \) and is parallel to
\[
\mathbf{u}_1 = i + 2j - k
\]
and a second line, \( L_2 \), passes through \( Q(-5, 2, 5) \) and is parallel to
\[
\mathbf{u}_2 = -4i + 4j + k.
\]

(a) Write down the vector equations for \( L_1 \) and \( L_2 \).

(b) Show that the lines \( L_1 \) and \( L_2 \) intersect and find the point of intersection.

(c) Determine the equation of the plane containing \( L_1 \) and \( L_2 \).

**Answers:**

\[
\begin{array}{c|c|c|c}
\text{a} & \mathbf{u}_1 = i + 2j - k & \text{direction vector} & \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\
\textbf{u}_2 = -4i + 4j + k & \text{direction vector} & \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 4 \\
1 \\
1 \\
\end{pmatrix} + \lambda \begin{pmatrix} 1 \\
2 \\
-1 \\
\end{pmatrix}, & \begin{pmatrix} -5 \\
2 \\
5 \\
\end{pmatrix} + \mu \begin{pmatrix} -4 \\
4 \\
1 \\
\end{pmatrix} & \cdot^1 & \cdot^2 & \text{for vector equations}^{1,2,3,6,8}.
\end{array}
\]
If they intersect

\[ \begin{align*}
2 + \lambda &= -5 - 4\mu \\
4 + 2\lambda &= 2 + 4\mu \\
1 - \lambda &= 5 + \mu \\
\end{align*} \]

\[\begin{align*}
4\mu + \lambda &= -7 \\
4\mu - 2\lambda &= 2 \\
\lambda &= -3 \\
\mu &= -1 \\
z_1 &= 1 - (-3) \\
z_2 &= 5 + (-1) \\
&= 4 \\
&= 4 \\
\end{align*}\]

Since \( z_1 = z_2 \), the lines intersect at \((-1, -2, 4)\).

(c) \( u_1 \times u_2 \) to get normal

\[
\begin{vmatrix}
i & j & k \\
1 & 2 & -1 \\
-4 & 4 & 1 \\
\end{vmatrix}
= i(2 + 4) - j(1 - 4) + k(4 + 8)
= 6i + 3j + 12k
\]

\[6x + 3y + 12z = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} \text{ (Point of intersection)} = 36\]

So equation of plane is \(6x + 3y + 12z = 36\)

OR \(2x + y + 4z = 12\)
Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $u$, $v$ and $w$ where

$$u = 5i + 13j, \quad v = 2i + j + 3k, \quad w = i + 4j - k.$$ 

Calculate $u \cdot (v \times w)$.

Interpret your result geometrically.

**Answers:**

$$\begin{vmatrix}
i & j & k \\
2 & 1 & 3 \\
1 & 4 & -1
\end{vmatrix} = -13i + 5j + 7k$$

$u \cdot (v \times w) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix} = 0$

$u$ lies in the same plane as the one containing both $v$ and $w$.

OR $u$ is parallel to the plane containing $v$ and $w$.

OR $u$ is perpendicular to the normal of $v$ and $w$.

OR All 4 points lie in the same plane.

OR $u$ is perpendicular to $v \times w$.

OR Volume of parallelepiped is zero.

OR $u$, $v$ and $w$ are coplanar/linearly dependent.

OR $u \cdot (v \times w) = \begin{vmatrix} 5 & 13 & 0 \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 3 & 0 \\ 4 & -1 & 3 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 0 \\ 1 & -1 & 3 \\ 1 & 4 & 1 \end{vmatrix} = 0$$

**1** Setting up combined product correctly.

**2** Correctly processes determinant.

**3** Correctly evaluates determinant to reach 0.
(8) (a) Find an equation of the plane \( \pi_1 \), through the points \( A(0, -1, 3) \), \( B(1, 0, 3) \) and \( C(0, 0, 5) \).

(b) \( \pi_2 \) is the plane through \( A \) with normal in the direction \(-j + k\).

Find an equation of the plane \( \pi_2 \).

(c) Determine the acute angle between planes \( \pi_1 \) and \( \pi_2 \).

**Answers:**

<p>| | | | | | |</p>
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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>( \overrightarrow{AB} = \begin{pmatrix} 1 \ 1 \ 0 \end{pmatrix} )</td>
<td>( \overrightarrow{AC} = \begin{pmatrix} 0 \ 1 \ 2 \end{pmatrix} )</td>
<td>OR</td>
<td>( \overrightarrow{BC} = \begin{pmatrix} -1 \ 0 \ 2 \end{pmatrix} )</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>( \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i &amp; j &amp; k \ 1 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 2 \end{vmatrix} )</td>
<td>= 2i – 2j + k</td>
<td>2x – 2y + z = 2 \times 0 – 2 \times -1 + 1 \times 3</td>
<td>( \pi_1: 2x - 2y + z = 5 )</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>OR</td>
<td>( r = \begin{pmatrix} 0 \ -1 \ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \ 1 \ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \ 1 \ 2 \end{pmatrix} )</td>
<td>or equivalent</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>0 \times 0 + (-1) \times (-1) + 1 \times 3 = 4</td>
<td>( \pi_2: -y + z = 4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Any two correct vectors.²
- Evidence of appropriate method.³
- Obtains vector product (any form).
- Obtains constant and states equation of plane.
- Evidence of appropriate method.⁴
- Processes to obtain equation of second plane.⁶
Normal vectors:

\[ n_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad n_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad |n_1| = \sqrt{9} = 3, \quad |n_2| = \sqrt{2} \]

\[
\cos (\text{angle between normals}) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{2 \times 0 - 2 \times -1 + 1 \times 1}{3 \sqrt{2}} = \frac{3}{3 \sqrt{2}} = \frac{1}{\sqrt{2}}
\]

Angle = 45°

The acute angle between planes is 45° (or \( \frac{\pi}{4} \)).

OR

\[ 2i - 2j + k, \quad \text{so} \quad |2i - 2j + k| = 3 \quad \text{and} \quad |-j + k| = \sqrt{2} \]

\[
3 = |n_1| \cdot |n_2| \cdot \cos \theta = 3 \sqrt{2} \cdot \cos \theta
\]

\[
\cos \theta = \frac{1}{\sqrt{2}} \quad \text{so} \quad \theta = \frac{\pi}{4} \quad (\text{or} \ 45°)
\]
Obtain an equation for the plane passing through the points $P(-2, 1, -1)$, $Q(1, 2, 3)$ and $R(3, 0, 1)$.

**Answer:**

*Method 1*

A normal to the plane:

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -2 & -2 & -2 \end{vmatrix}$$

$$= i \begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -2 & -2 \end{vmatrix} + k \begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix}$$

$$= 6\mathbf{i} + 14\mathbf{j} - 8\mathbf{k}$$

Hence the equation has the form:

$$6x + 14y - 8z = d.$$  

The plane passes through $P(-2, 1, -1)$ so

$$d = -12 + 14 + 8 = 10$$

which gives an equation $6x + 14y - 8z = 10$, i.e. $3x + 7y - 4z = 5$.

*Method 2*

A plane has an equation of the form

$$ax + by + cz = d.$$ Using the points $P, Q, R$ we get

$$-2a + b - c = d$$

$$a + 2b + 3c = d$$

$$3a + c = d$$

Using Gaussian elimination to solve these we have

$$\begin{vmatrix} -2 & 1 & -1 & d \\ 1 & 2 & 3 & d \\ 3 & 0 & 1 & d \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 5 & 5 & 3d \\ 6 & 8 & 2d \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 0 & 5 & 5 & 3d \\ 0 & 0 & 2 & \frac{-8d}{5} \end{vmatrix}$$

$$\Rightarrow c = \frac{4}{5}d, \quad b = \frac{7}{5}d, \quad a = \frac{3}{5}d$$

These give the equation

$$\left(\frac{3}{5}d\right)x + \left(\frac{7}{5}d\right)y + \left(-\frac{4}{5}d\right)z = d$$

i.e. $3x + 7y - 4z = 5$. 

**Note:** the method used in the answer is correct, but the working could be clearer and more concise. The first method is also correct, but the final equation can be simplified further.
The lines $L_1$ and $L_2$ are given by the equations
\[
\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \quad \text{and} \quad \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},
\]
respectively.

Find:

(a) the value of $k$ for which $L_1$ and $L_2$ intersect and the point of intersection;

(b) the acute angle between $L_1$ and $L_2$.

Answers:

(a) In terms of a parameter $s$, $L_1$ is given by
\[
x = 1 + ks, \quad y = -s, \quad z = -3 + s
\]

In terms of a parameter $t$, $L_2$ is given by
\[
x = 4 + t, \quad y = -3 + t, \quad z = -3 + 2t
\]

Equating the $y$ coordinates and equating the $z$ coordinates:
\[
\begin{align*}
-s &= -3 + t \\
-3 + s &= -3 + 2t
\end{align*}
\]

Adding these
\[
-3 = -6 + 3t \\
\Rightarrow t = 1 \Rightarrow s = 2.
\]

From the $x$ coordinates
\[
1 + ks = 4 + t
\]
Using the values of $s$ and $t$
\[
1 + 2k = 5 \Rightarrow k = 2
\]

The point of intersection is: $(5, -2, -1)$.

(b) $L_1$ has direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.  
$L_2$ has direction $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.  

For both directions.

Let the angle between $L_1$ and $L_2$ be $\theta$, then
\[
\cos \theta = \frac{(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}
\]
\[
\theta = 60^\circ
\]

The angle between $L_1$ and $L_2$ is $60^\circ$. 
Given $u = -2i + 5k$, $v = 3i + 2j - k$ and $w = -i + j + 4k$.
Calculate $u \cdot (v \times w)$. 

\[
\begin{align*}
\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} & \quad 1M \\
= i \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - j \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + k \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & \quad 1 \\
= 9i - 11j + 5k & \quad 1 \\
\end{align*}
\]

Thus, $u \cdot (v \times w) = (-2i + 0j + 5k). (9i - 11j + 5k)$

\[
= -18 + 0 + 25 \\
= 7. \\
\]

Answer:

\[
\begin{align*}
\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} & \quad 1M \\
= i \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - j \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + k \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & \quad 1 \\
= 9i - 11j + 5k & \quad 1 \\
\end{align*}
\]

\[
\begin{align*}
\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix} & \quad 1M \\
= i \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - j \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + k \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & \quad 1 \\
= 9i - 11j + 5k & \quad 1 \\
\end{align*}
\]
(a) Use Gaussian elimination to solve the following system of equations

\[
\begin{align*}
    x + y - z &= 6 \\
    2x - 3y + 2z &= 2 \\
    -5x + 2y - 4z &= 1.
\end{align*}
\]

(b) Show that the line of intersection, \( L \), of the planes \( x + y - z = 6 \) and \( 2x - 3y + 2z = 2 \) has parametric equations

\[
\begin{align*}
    x &= \lambda \\
    y &= 4\lambda - 14 \\
    z &= 5\lambda - 20.
\end{align*}
\]

(c) Find the acute angle between line \( L \) and the plane \(-5x + 2y - 4z = 1\).
(b) Let \( x = \lambda \).

**Method 1**

In first plane: \( x + y - z = 6 \).
\[
\lambda + (4\lambda - 14) - (5\lambda - 20) = 5\lambda - 5\lambda + 6 = 6.
\]

In the second plane:
\[
2x - 3y + 2z = 2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 5\lambda - 5\lambda + 2 = 2.
\]

**Method 2**

\[
y - z = 6 - \lambda \quad \Rightarrow \quad y = 6 + z - \lambda \\
-3y + 2z = 2 - 2\lambda
\]

\[
(-18 - 3z + 3\lambda) + 2z = 2 - 2\lambda
\]

\[
-\lambda = 20 - 5\lambda \quad \Rightarrow \quad z = 5\lambda - 20
\]

and \( y = 4\lambda - 14 \)

**Method 2**

\[
x + y - z = 6 \quad \quad (1)
\]
\[
2x - 3y + 2z = 2 \quad \quad (2)
\]
\[
5x - z = 20 \quad (2) + 3(1)
\]
\[
4x - y = 14 \quad (2) + 2(1)
\]

\[
y = 4x - 14
\]
\[
z = 5x - 20
\]

\[
x = \lambda, \ y = 4\lambda - 14, \ z = 5\lambda - 20
\]

(c) Direction of \( L \) is \( \mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \), direction of normal to the plane is \(-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \). Letting \( \theta \) be the angle between these then

\[
\cos \theta = \frac{-5 + 8 - 20}{\sqrt{42\sqrt{45}}}
\]

\[
= \frac{-17}{3\sqrt{210}}
\]

This gives a value of 113.0° which leads to the angle

\[
113.0° - 90° = 23.0°.
\]
(a) Find an equation of the plane \( \pi_1 \) through the points \( A(1, 1, 1) \), \( B(2, -1, 1) \) and \( C(0, 3, 3) \).

(b) The plane \( \pi_2 \) has equation \( x + 3y - z = 2 \).

Given that the point \( (0, a, b) \) lies on both the planes \( \pi_1 \) and \( \pi_2 \), find the values of \( a \) and \( b \). Hence find an equation of the line of intersection of the planes \( \pi_1 \) and \( \pi_2 \).

(c) Find the size of the acute angle between the planes \( \pi_1 \) and \( \pi_2 \).

Answers:

\[ \overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} \quad \overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \]

\[ \overrightarrow{AB} \times \overrightarrow{AC} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{array} \right| = (-4 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (2 - 2)\mathbf{k} \]

\[ = -4\mathbf{i} - 2\mathbf{j} \]

Equation is

\[ -4x - 2y = k \]

\[ = -4(1) - 2(1) = -6 \]

\[ \text{i.e. } -2x - y = -3 \]

\[ 2x + y = 3 \]

PTO for (b) & (c)
(b) In \( \pi_1 \): \( 2 \times 0 + a = 3 \Rightarrow a = 3 \).
In \( \pi_2 \): \( 0 + 3a - b = 2 \Rightarrow b = 3a - 2 = 7 \).
Hence the point of intersection is (0, 3, 7).
Line of intersection: direction from
\[
\begin{vmatrix}
   i & j & k \\
  -4 & -2 & 0 \\
   1 & 3 & -1
\end{vmatrix} = 2i - 4j - 10k
\]
\[x = 0 + 2t; \ y = 3 - 4t; \ z = 7 - 10t\]

There are many valid variations on this (including symmetric form) and these were marked on their merits.

(c) Let the angle be \( \theta \), then
\[
\cos \theta = \frac{|(-4i - 2j) \cdot (i + 3j - k)|}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}} = \frac{-4 - 6}{\sqrt{20 \times 11}} = \frac{5}{\sqrt{55}} \quad \text{1M, 1}
\]
or
\[
\sin \theta = \frac{|(-4i - 2j) \times (i + 3j - k)|}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}} \quad \text{1M}
\]
\[
= \frac{\sqrt{2^2 + 4^2 + 10^2}}{\sqrt{20\sqrt{11}}} = \frac{\sqrt{120}}{20 \times 11} = \sqrt{\frac{6}{11}} \quad \text{1}
\]
Hence \( \theta \approx 47.6^\circ \).
Lines $L_1$ and $L_2$ are given by the parametric equations

$\begin{align*}
L_1 : x &= 2 + s, \quad y = -s, \quad z = 2 - s \\
L_2 : x &= -1 - 2t, \quad y = t, \quad z = 2 + 3t.
\end{align*}$

(a) Show that $L_1$ and $L_2$ do not intersect.

(b) The line $L_3$ passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both $L_1$ and $L_2$. Obtain parametric equations for $L_3$.

(c) Find the coordinates of the point $Q$ where $L_3$ and $L_2$ intersect and verify that $P$ lies on $L_1$.

(d) $PQ$ is the shortest distance between the lines $L_1$ and $L_2$. Calculate $PQ$.

Answers:

(a) Equating the $x$-coordinates: $2 + s = -1 - 2t \Rightarrow s + 2t = -3 \ (1)$
EQUATING THE $y$-COORDINATES: $-s = t \Rightarrow s = -t$

(b) Substituting in (1): $-t + 2t = -3 \Rightarrow t = -3 \Rightarrow s = 3$.
Putting $s = 3$ in $L_1$ gives $(5, -3, -1)$ and $t = -3$ in $L_2$, $(5, -3, -7)$.
As the $z$ coordinates differ, $L_1$ and $L_2$ do not intersect.

(b) Directions of $L_1$ and $L_2$ are: $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The vector product of these gives the direction of $L_3$.

$$
(i - j - k) \times (-2i + j + 3k) = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2i - j - k \quad \text{1M,1}
$$

Equation of $L_3$:

$$
r = i + j + 3k + (-2i - j - k)u
= (1 - 2u)i + (1 - u)j + (3 - u)k
$$

Hence $L_3$ is given by $x = 1 - 2u, y = 1 - u, z = 3 - u$.  \[1\]

(c) Solving the $x$ and $y$ coordinates of $L_3$ and $L_2$:

$$-1 - 2t = 1 - 2u \quad \text{and} \quad t = 1 - u$$

$\Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0
$\[1\]

The point of intersection, $Q$, is $(-1, 0, 2)$ since $2 + 3t = 2$ and $3 - u = 2$.

$L_1$ is $x = 2 + s, y = -s, z = 2 - s$. When $x = 1, s = -1$ and hence

$y = 1$ and $z = 3$, i.e. $P$ lies on $L_1$.  \[1\]

(d) $PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$.  \[1\]