Methods of Proof

AH Maths Exam Questions

Source: 2019 Specimen P1 Q7 AH Maths – Same as 2018 Q12

Prove by induction that, for all positive integers \( n \),

\[
\sum_{r=1}^{n} 3^{r-1} = \frac{1}{2}(3^n - 1).
\]

Answer:

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
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<tbody>
<tr>
<td>1</td>
<td>Show true for ( n = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>Assume (statement) true for ( n = k ) AND consider whether (statement) true for ( n = k + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>Correct statement for sum to ((k+1)) terms using inductive hypothesis</td>
</tr>
<tr>
<td>4</td>
<td>Combine terms in ( 3^k )</td>
</tr>
<tr>
<td>5</td>
<td>Express sum explicitly in terms of ((k+1)) or achieve stated aim/goal AND communicate</td>
</tr>
<tr>
<td>1</td>
<td>LHS: ( 3^0 = 1 )  ( \text{RHS: } \frac{1}{2}(3-1)=1 )  So true for ( n = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>Suitable statement and ( \sum_{r=1}^{k} 3^{r-1} = \frac{1}{2}(3^k - 1) )  ( \text{AND } \sum_{r=1}^{k+1} 3^{r-1} = \ldots )</td>
</tr>
<tr>
<td>3</td>
<td>( \ldots = \frac{1}{2}(3^k - 1) + 3^{(k+1)-1} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{3}{2} \times 3^k - \frac{1}{2} )</td>
</tr>
</tbody>
</table>
| 5    | \( \frac{1}{2}(3^{(k+1)} - 1) \)  If true for \( n = k \) then true for \( n = k + 1 \). Also shown true for \( n = 1 \) therefore, by induction, true for all positive integers \( n \).


For each of the following statements, decide whether it is true or false. If true, give a proof; if false, give a counterexample.

A. If a positive integer $p$ is prime, then so is $2p + 1$.

B. If a positive integer $n$ has remainder 1 when divided by 3, then $n^3$ also has remainder 1 when divided by 3.

Answers:

1. give counterexample
   - for example choose $p = 7$
   - $2(7) + 1 = 15$, which is not prime.
   - $\therefore$ statement is false.

2. set up $n$
   - $n = 3a + 1$, $a \in \mathbb{W}$

3. consider expansion of $n^3$
   - $n^3 = 27a^3 + 27a^2 + 9a + 1$

4. complete proof with conclusion
   - $= 3(9a^3 + 9a^2 + 3a) + 1$ and statement such as “so $n^3$ has remainder 1 when divided by 3 : $\therefore$ statement is true”.
Let $n$ be a positive integer.

(a) Find a counterexample to show that the following statement is false.

$$n^2 + n + 1$$

is always a prime number.

(b) (i) Write down the contrapositive of:

If $n^2 - 2n + 7$ is even then $n$ is odd.

(ii) Use the contrapositive to prove that if $n^2 - 2n + 7$ is even then $n$ is odd.

**Answers:**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>state counterexample</td>
<td>1, 2</td>
<td>eg when $n = 4$, $n^2 + n + 1 = 21$ which is not prime</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th>(i)</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>write down contrapositive statement</td>
<td>1, 2, 8</td>
<td>If $n$ is even then $n^2 - 2n + 7$ is odd</td>
<td>1</td>
</tr>
</tbody>
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<thead>
<tr>
<th>(ii)</th>
<th></th>
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<tbody>
<tr>
<td>write down appropriate form for $n$ AND substitute</td>
<td>1, 3, 4, 5, 9</td>
<td>$n = 2k$, $k \in \mathbb{N}$ and $(2k)^2 - 2(2k) + 7$</td>
<td>3</td>
</tr>
<tr>
<td>show $n^2 - 2n + 7$ is odd</td>
<td>1, 6, 7, 9</td>
<td>eg $2(2k^2 - 2k + 3) + 1$ which is odd since $2k^2 - 2k + 3 \in \mathbb{N}$</td>
<td></td>
</tr>
<tr>
<td>communicate</td>
<td>1, 8, 9</td>
<td>contrapositive statement is true AND therefore original statement is true</td>
<td></td>
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</tbody>
</table>
Prove by induction that
\[ \sum_{r=1}^{n} r!r = (n+1)! - 1 \] for all positive integers \( n \).

<table>
<thead>
<tr>
<th>Generic scheme</th>
<th>Illustrative scheme</th>
<th>Max mark</th>
</tr>
</thead>
</table>
| 1 show true when \( n = 1 \) | 1 when \( n = 1 \)
LHS = 1! + 1 = 1
RHS = (1+1)! - 1 = 1
so result is true when \( n = 1 \). | 5 |
| 2 assume (statement) true for \( n = k \) AND consider whether (statement) true for \( n = k+1 \) | 2 suitable statement
AND \( \sum_{r=1}^{k} r!r = (k+1)! - 1 \)
AND \( \sum_{r=1}^{k+1} r!r = \ldots \) | |
| 3 state sum to \( (k+1) \) terms using inductive hypothesis | 3 \( (k+1)! - 1 + (k+1)! (k+1) \) | |
| 4 extract \( (k+1)! \) as common factor | 4 \( (k+1)!(k+2) - 1 \) | |
| 5 express sum explicitly in terms of \( (k+1) \) or achieve stated aim/goal AND communicate | 5 \( ((k+1)+1)! - 1 \)
AND
If true for \( n = k \) then true for \( n = k + 1 \). Also shown true for \( n = 1 \) therefore, by induction, true for all positive integers \( n \). | |
(5) Prove directly that:

(a) the sum of any three consecutive integers is divisible by 3;

(b) any odd integer can be expressed as the sum of two consecutive integers.

**Answers:**

<table>
<thead>
<tr>
<th>(a)</th>
<th>•¹ form the sum of three consecutive integers (_{1,2,3,4,5})</th>
<th>•¹ ((n - 1) + n + (n + 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>•² communication (_{1,5})</td>
<td>•² (3n) which is divisible by 3</td>
</tr>
</tbody>
</table>

| (b) | •³ appropriate form for odd number, decomposed into two consecutive integers \(_{1,2,3}\) | •³ \(2k + 1 = k + (k + 1)\), \(k \in \mathbb{Z}\) |
Let $n$ be an integer.

Using proof by contrapositive, show that if $n^2$ is even, then $n$ is even.

Answer:

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<tr>
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<th>Max mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdot^1$ write down contrapositive statement $^{1,2,7,8}$</td>
<td>$\cdot^1$ The contrapositive of the original statement is: If $n$ is odd then $n^2$ is odd</td>
<td>4</td>
</tr>
<tr>
<td>$\cdot^2$ write down appropriate form for $n$ $^{3,4,7}$</td>
<td>$\cdot^2$ $n = 2k + 1$ , $k \in \mathbb{Z}$</td>
<td></td>
</tr>
<tr>
<td>$\cdot^3$ show $n^2$ is odd $^{5,6,7}$</td>
<td>$\cdot^3$ $n^2 = 2(2k^2 + 2k) + 1$ which is odd</td>
<td></td>
</tr>
<tr>
<td>$\cdot^4$ communicate</td>
<td>$\cdot^4$ contrapositive statement is true therefore original statement is true</td>
<td></td>
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</tbody>
</table>
(7) Prove by induction that

\[ \sum_{r=1}^{n} r(3r-1) = n^2(n+1), \quad \forall n \in \mathbb{N}. \]

**Answer:**

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<th>Generic Scheme</th>
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<th>Max Mark</th>
</tr>
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<tbody>
<tr>
<td>1 show true for ( n = 1 )</td>
<td>( \text{LHS: } 1(3-1) = 2 ) \text{ RHS: } 1^2(1+1) = 2 \text{ So true for } n = 1</td>
<td>4</td>
</tr>
<tr>
<td>2 assume true for ( n = k ) AND consider ( n = k + 1 )</td>
<td>( \sum_{r=1}^{k} r(3r-1) = k^2(k+1) ) AND ( \sum_{r=1}^{k+1} r(3r-1) = ) ( \cdots = \sum_{r=1}^{k} r(3r-1) + (k+1)(3(k+1)-1) )</td>
<td>2</td>
</tr>
<tr>
<td>3 correct statement of sum to ( (k+1) ) terms using inductive hypothesis</td>
<td>( = k^2(k+1) + (k+1)(3k+2) ) ( = (k+1)[k^2 + 3k + 2] ) ( = (k+1)(k+1)(k+2) )</td>
<td>3,4</td>
</tr>
<tr>
<td>4 express explicitly in terms of ( (k+1) ) or achieve stated aim/goal ( \text{AND communicate} )</td>
<td>( = (k+1)^3((k+1)+1) ), thus if true for ( n = k ) then true for ( n = k + 1 ) but since true for ( n = 1 ), then by induction true for all ( n \in \mathbb{N} )</td>
<td>3,4</td>
</tr>
</tbody>
</table>
Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8.

<table>
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<tr>
<td>Let numbers be $2n - 1, 2n + 1, \quad n \in \mathbb{N}$</td>
</tr>
<tr>
<td>$(2n + 1)^2 - (2n - 1)^2$</td>
</tr>
<tr>
<td>$= (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$</td>
</tr>
<tr>
<td>$= 8n$ which is divisible by 8</td>
</tr>
</tbody>
</table>

3 - 1. correct form for any two consecutive odd numbers $1^2$.  
2. correct expressions squared out.  
3. multiple of 8 and communication.
Given $A$ is the matrix \[
\begin{pmatrix}
2 & a \\
0 & 1
\end{pmatrix},
\]
prove by induction that
\[
A^n = \begin{pmatrix}
2^n & a(2^n - 1) \\
0 & 1
\end{pmatrix}, \quad n \geq 1.
\]

<table>
<thead>
<tr>
<th>Expected Answer/s</th>
<th>Max Mark</th>
<th>Additional Guidance</th>
</tr>
</thead>
</table>
| For $n = 1$ RHS = \[
\begin{pmatrix}
2 & a \cdot 2^1 - 1 \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2 & a \\
0 & 1
\end{pmatrix}
\]
\[
= A
\]
LHS = $A^1 = A = $RHS. 
Assume true for $n = k$,
\[
A^k = \begin{pmatrix}
2^k & a \cdot 2^k - 1 \\
0 & 1
\end{pmatrix}
\]
Consider $n = k + 1$,
\[
A^{k+1} = A^k A^1 \quad \text{[OR $A^1 A^k$]}
\]
\[
= \begin{pmatrix}
2^k & a \cdot 2^k - 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
2 & a \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2^k \cdot 2 & 2^k a + a \cdot 2^k - 1 \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2^{k+1} & 2 \cdot 2^k a + a \cdot 2^k - a \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2^{k+1} & a(2^{k+1} - 1) \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2^{k+1} & a(2^{k+1} - 1) \\
0 & 1
\end{pmatrix}
\]
Hence, if true for $n = k$, then true for $n = k + 1$, but since true for $n = 1$, then by induction true for all positive integers $n$. 

1. Substituting $n = 1$.

2. Inductive hypothesis (must include "Assume true for $n = k$..." or equivalent phrase) and expansion of $A^{k+1}$.

3. Correct multiplication of two matrices and accurate manipulation of indices and brackets.

4. Line $*$ and statement of result in terms of $(k + 1)$ and valid statement of conclusion. 

5. See note 5.

6. 
Prove by induction that, for all positive integers \( n \),

\[
\sum_{r=1}^{n} \left( 4r^3 + 3r^2 + r \right) = n(n + 1)^3
\]

**Answer:**

For \( n = 1 \)

\[
\begin{align*}
\text{L.H.S} & \quad \text{R.H.S} \\
\sum_{r=1}^{n} \left( 4r^3 + 3r^2 + r \right) & \quad n(n + 1)^3 \\
= 4 + 3 + 1 & \quad = 1 \times 2^3 \\
= 8 & \quad = 8 \\
\Rightarrow \text{true for } n = 1
\end{align*}
\]

Assume true for \( n = k \),

\[
\sum_{r=1}^{k} \left( 4r^3 + 3r^2 + r \right) = k(k + 1)^3
\]

Consider \( n = k + 1 \),

\[
\begin{align*}
\sum_{r=1}^{k+1} \left( 4r^3 + 3r^2 + r \right) & = \sum_{r=1}^{k} \left( 4r^3 + 3r^2 + r \right) + 4(k + 1)^3 + 3(k + 1)^2 + (k + 1) \\
& = k(k + 1)^3 + 4(k + 1)^3 + 3(k + 1)^2 + (k + 1) \\
& = (k + 1) \left[ k(k + 1)^2 + 4(k + 1)^2 + 3(k + 1) + 1 \right] \\
& = (k + 1) \left[ k(k^2 + 2k + 1) + 4(k^2 + 2k + 1) + 3(k + 1) + 1 \right] \\
& = (k + 1) \left[ k^3 + 2k^2 + k + 4k^2 + 8k + 4 + 3k + 3 + 1 \right] \\
& = (k + 1) \left( k^3 + 6k^2 + 12k + 8 \right) \\
& = (k + 1)(k + 2)^3 \\
& = (k + 1)(k + 1 + 1)^3
\end{align*}
\]

Hence, if true for \( n = k \), then true for \( n = k + 1 \), but since true for \( n = 1 \), then by induction true for all positive integers \( n \).
Let \( n \) be a natural number. For each of the following statements, decide whether it is true or false. If true, give a proof; if false, give a counterexample.

| \( A \) | If \( n \) is a multiple of 9 then so is \( n^2 \). |
| \( B \) | If \( n^2 \) is a multiple of 9 then so is \( n \). |

**Answer:**

| \( A \) | Suppose \( n = 9m \) for some natural number [positive integer], \( m \). |
| \( A \) | Then \( n^2 = 81m^2 = 9(9m^2) \) |
| \( A \) | Hence \( n^2 \) is a multiple of 9, so \( A \) is true. |
| \( B \) | **False.** Accept any valid counterexample: \( n = 3, 6, 12, 15, 21 \) etc |
| \( 1 \) | Generalisation, using different letter. |
| \( 2 \) | Correct multiplication and 9 extracted as a factor. |
| \( 3 \) | Conclusion of proof and state \( A \) true. |
| \( 4 \) | Valid counterexample and conclusion. |
(a) Prove by induction that

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\]

for all integers \(n \geq 1\).

(b) Show that the real part of \(\frac{(\cos \frac{\pi}{18} + i \sin \frac{\pi}{18})^{11}}{(\cos \frac{\pi}{36} + i \sin \frac{\pi}{36})^{11}}\) is zero.

Answers:

(a) For \(n = 1\), the LHS \(= \cos \theta + i \sin \theta\) and the RHS \(= \cos \theta + i \sin \theta\). Hence the result is true for \(n = 1\).

Assume the result is true for \(n = k\), i.e.

\[(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta.\]

1 working with \(n\) is penalised.

Now consider the case when \(n = k + 1\):

\[(\cos \theta + i \sin \theta)^{k+1} = (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta)\]

1 for applying the inductive hypothesis

1 multiplying and collecting

\[= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)\]

\[= \cos(k + 1)\theta + i \sin(k + 1)\theta\]

Thus, if the result is true for \(n = k\) the result is true for \(n = k + 1\).

Since it is true for \(n = 1\), the result is true for all \(n \geq 1\).

(b) \(\frac{(\cos \frac{\pi}{18} + i \sin \frac{\pi}{18})^{11}}{(\cos \frac{\pi}{36} + i \sin \frac{\pi}{36})^{11}} = \frac{\cos \frac{11\pi}{18} + i \sin \frac{11\pi}{18}}{\cos \frac{\pi}{9} + i \sin \frac{\pi}{9}}\)

1 using result from above

\[= \frac{\cos \frac{11\pi}{18} \cos \frac{\pi}{9} + \sin \frac{11\pi}{18} \sin \frac{\pi}{9}}{\cos \frac{\pi}{9} + \sin \frac{\pi}{9}} + \text{imaginary term}\]

\[= \cos \left(\frac{11\pi}{18} - \frac{\pi}{9}\right) + \text{imaginary term}\]

\[= \cos \frac{\pi}{2} + \text{imaginary term}\]

Thus the real part is zero as required. 1 or equivalent
Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 for all integers $n \geq 2$.

Answer:

For $n = 2$, $8^2 + 3^0 = 64 + 1 = 65$. True when $n = 2$. 1

Assume true for $k$, i.e. that $8^k + 3^{k-2}$ is divisible by 5, i.e. can be expressed as $5p$ for an integer $p$. 1 for the inductive hypothesis

Now consider $8^{k+1} + 3^{k-1}$

$= 8 \times 8^k + 3^{k-1}$ 1

$= 8 \times (5p - 3^{k-2}) + 3^{k-1}$ 1 for replacing $8^k$

$= 40p - 3^{k-2}(8 - 3)$

$= 5(8p - 3^{k-2})$ which is divisible by 5. 1

So, since it is true for $n = 2$, it is true for all $n \geq 2$.
(14)  
(a) Prove that the product of two odd integers is odd.  
(b) Let \( p \) be an odd integer. Use the result of (a) to prove by induction that \( p^n \) is odd for all positive integers \( n \).

Answers:

(a) Write the odd integers as: \( 2n + 1 \) and \( 2m + 1 \) where \( n \) and \( m \) are integers.  
Then  
\[
(2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 \\
= 2(2nm + n + m) + 1 
\]
which is odd.

(b) Let \( n = 1, p^1 = p \) which is given as odd.  
Assume \( p^k \) is odd and consider \( p^{k+1} \).  
\[
p^{k+1} = p^k \times p
\]
Since \( p^k \) is assumed to be odd and \( p \) is odd, \( p^{k+1} \) is the product of two odd integers is therefore odd.  
Thus \( p^{n+1} \) is an odd integer for all \( n \) if \( p \) is an odd integer.

Source: 2010 Q12 AH Maths

(15) Prove by contradiction that if \( x \) is an irrational number, then \( 2 + x \) is irrational.

Answer:

Assume \( 2 + x \) is rational  
and let \( 2 + x = \frac{p}{q} \) where \( p, q \) are integers.  

So  
\[
x = \frac{p}{q} - 2 \\
= \frac{p - 2q}{q}
\]
Since \( p - 2q \) and \( q \) are integers, it follows that \( x \) is rational. This is a contradiction.
Prove by induction that, for all positive integers $n,$

$$
\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.
$$

**Answer:**

When $n = 1,$ LHS $= \frac{1}{1 \times 2} = \frac{1}{2},$ RHS $= 1 - \frac{1}{2} = \frac{1}{2}.$ So true when $n = 1.$

Assume true for $n = k,$ $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k + 1}.$

Consider $n = k + 1$

$$
\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}
$$

$$
= 1 - \frac{1}{k + 1} + \frac{1}{(k+1)(k+2)}
$$

$$
= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k + 1)((k + 1) + 1)}
$$

$$
= 1 - \frac{1}{((k + 1) + 1)}
$$

Thus, if true for $n = k,$ statement is true for $n = k + 1,$ and, since true for $n = 1,$ true for all $n \geq 1.$
### Source: 2008 Q11 AH Maths

<table>
<thead>
<tr>
<th>(17)</th>
<th>For each of the following statements, decide whether it is true or false and prove your conclusion.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>For all natural numbers $m$, if $m^2$ is divisible by 4 then $m$ is divisible by 4.</td>
</tr>
<tr>
<td>B</td>
<td>The cube of any odd integer $p$ plus the square of any even integer $q$ is always odd.</td>
</tr>
</tbody>
</table>

### Answers:

(a) **Counter example $m = 2$.**
So statement is false.  

(b) **Let the numbers be $2n + 1$ and $2m$ then**

\[
(2n + 1)^3 + (2m)^2 = 8n^3 + 12n^2 + 6n + 1 + 4m^2
\]

\[
= 2(4n^3 + 6n^2 + 3n + 2m^2) + 1
\]

which is odd.

**OR**

Proving algebraically that either the cube of an odd number is odd or the square of an even number is even.

Odd cubed is odd and even squared is even.

So the sum of them is odd.
Prove by induction that for \( a > 0 \),
\[
(1 + a)^n \geq 1 + na
\]
for all positive integers \( n \).

### Answers:

Consider \( n = 1 \), LHS = \((1 + a)\), RHS = \(1 + a\) so true for \( n = 1 \).

Assume that \((1 + a)^k \geq 1 + ka\) and consider \((1 + a)^{k+1}\).
\[
(1 + a)^{k+1} = (1 + a)(1 + a)^k \\
\geq (1 + a)(1 + ka) \\
= 1 + a + ka + ka^2 \\
= 1 + (k + 1)a + ka^2 \\
> 1 + (k + 1)a \text{ since } ka^2 > 0
\]
as required. So since true for \( n = 1 \), by mathematical induction statement
is true for all \( n \geq 1 \).