Further Sequences & Series (Maclaurin Expansion)
AH Maths Exam Questions

Source: 2018 Q17 AH Maths

(1)

(a) Given $f(x) = e^{2x}$, obtain the Maclaurin expansion for $f(x)$ up to, and including, the term in $x^3$.

(b) On a suitable domain, let $g(x) = \tan x$.
   (i) Show that the third derivative of $g(x)$ is given by $g'''(x) = 2\sec^4 x + 4\tan^2 x\sec^2 x$.
   (ii) Hence obtain the Maclaurin expansion for $g(x)$ up to and including the term in $x^3$.

(c) Hence, or otherwise, obtain the Maclaurin expansion for $e^{2x}\tan x$ up to, and including, the term in $x^3$.

(d) Write down the first three non-zero terms in the Maclaurin expansion for $2e^{2x}\tan x + e^{2x}\sec^2 x$.

Answers:

(a) $f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 \ldots$

(b) (i) $g'''(x) = 2\sec^4 x + 4\tan^2 x\sec^2 x + (4\sec^2 xtan x)\tan x$
   (ii) $g(x) = x + \frac{1}{3}x^3 \ldots$

(c) $x + 2x^2 + \frac{7}{3}x^3 \ldots$

(d) $1 + 4x + 7x^2$
### Source: 2016 Q6 AH Maths

<table>
<thead>
<tr>
<th>(2)</th>
<th>Find Maclaurin expansions for ( \sin 3x ) and ( e^{4x} ) up to and including the term in ( x^3 ). Hence obtain an expansion for ( e^{4x} \sin 3x ) up to and including the term in ( x^3 ).</th>
</tr>
</thead>
</table>

#### Answers:

\[
f(x) = 3x - \frac{9}{2}x^3, \quad f(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3
\]

\[
3x + 12x^2 + \frac{39}{2}x^3 + \ldots
\]

### Source: 2014 Q9 AH Maths

<table>
<thead>
<tr>
<th>(3)</th>
<th>Give the first three non-zero terms of the Maclaurin series for ( \cos 3x ). Write down the first four terms of the Maclaurin series for ( e^{2x} ). Hence, or otherwise, determine the Maclaurin series for ( e^{2x} \cos 3x ) up to, and including, the term in ( x^3 ).</th>
</tr>
</thead>
</table>

#### Answers:

\[
\cos 3x = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} \ldots
\]

\[
e^{2x} \cos 3x = 1 + 2x - \frac{5x^2}{2} - \frac{23x^3}{3}
\]

### Source: 2012 Q6 AH Maths

<table>
<thead>
<tr>
<th>(4)</th>
<th>Write down the Maclaurin expansion of ( e^{x} ) as far as the term in ( x^3 ). Hence, or otherwise, obtain the Maclaurin expansion of ( (1 + e^{x})^2 ) as far as the term in ( x^3 ).</th>
</tr>
</thead>
</table>

#### Answers:

\[
e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
\]

\[
(1 + e^{x})^2 = 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots
\]
### Source: 2010 Q9 AH Maths

<table>
<thead>
<tr>
<th>(5)</th>
<th>Obtain the first three non-zero terms in the Maclaurin expansion of ((1 + \sin^2 x)).</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Answer:</strong></td>
<td>(f(x) = 1 + x^2 - \frac{x^4}{3} + \ldots)</td>
</tr>
</tbody>
</table>

### Source: 2009 Q14 AH Maths

| (6) | Express \(\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)}\) in partial fractions.  
Hence, or otherwise, obtain the first three non-zero terms in the Maclaurin expansion of \(\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)}\). |
|-----|---------------------------------------------------------------------------------------------------------------|
| **Answers:** | \[
\frac{2}{(x + 2)^2} + \frac{1}{x - 4} \quad 
\frac{1}{4} - \frac{9x}{16} + \frac{23x^2}{64} + \ldots
\] |
Obtain the first four terms in the Maclaurin series of $\sqrt{1+x}$, and hence write down the first four terms in the Maclaurin series of $\sqrt{1+x^2}$.

Hence obtain the first four terms in the Maclaurin series of $\sqrt{(1+x)(1+x^2)}$.

Answers:

Let $f(x) = (1 + x)^{\frac{1}{2}}$, then

\[
\begin{align*}
  f(x) &= (1 + x)^{\frac{1}{2}} \Rightarrow f(0) = 1 \\
  f'(x) &= \frac{1}{2} (1 + x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2} \\
  f''(x) &= -\frac{1}{4} (1 + x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4} \\
  f'''(x) &= \frac{3}{8} (1 + x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}
\end{align*}
\]

Hence

\[
\sqrt{1+x} = 1 + \frac{1}{2} x - \frac{1}{4} \times \frac{x^2}{2} + \frac{3}{8} \times \frac{x^3}{6} - \ldots
\]

\[
= 1 + \frac{1}{2} x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots
\]

and replacing $x$ by $x^2$ gives

\[
\sqrt{1 + x^2} = 1 + \frac{1}{2} x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots
\]

Thus

\[
\sqrt{(1 + x)(1 + x^2)} = 
\left( 1 + \frac{1}{2} x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots \right) \left( 1 + \frac{1}{2} x^2 - \frac{x^4}{8} + \frac{x^5}{16} - \ldots \right)
\]

for multiplying

\[
= 1 + \frac{1}{2} x + \frac{1}{2} x^2 - \frac{1}{8} x^2 + \frac{1}{4} x^3 + \frac{1}{16} x^3 + \ldots
\]

\[
= 1 + \frac{1}{2} x + \frac{3}{8} x^2 + \frac{5}{16} x^3 + \ldots
\]
### 2008 Q12 AH Maths

| (8) | Obtain the first three non-zero terms in the Maclaurin expansion of $x \ln(2 + x)$. Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of $x \ln(2 - x)$. Hence obtain the first \textbf{two} non-zero terms in the Maclaurin expansion of $x \ln(4 - x^2)$. 

*Throughout this question, it can be assumed that $-2 < x < 2$.** |

| **Answers:** |

First three non-zero terms: $f(x) = \ln(2) x + \frac{x^2}{2} - \frac{x^3}{8} + \cdots$

$\ln(2 - x) = (\ln 2) x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$

$\ln(4 - x^2) = (2\ln 2) x - \frac{x^3}{4} + \cdots$

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### 2007 Q6 AH Maths

| (9) | Find the Maclaurin series for $\cos x$ as far as the term in $x^4$. Deduce the Maclaurin series for $f(x) = \frac{1}{2} \cos 2x$ as far as the term in $x^4$. Hence write down the first three non-zero terms of the series for $f(3x)$. |

| **Answers:** |

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$

$f(x) = \frac{1}{2} - x^2 + \frac{x^4}{3} - \cdots$

$f(3x) = \frac{1}{2} - 9x^2 + 24x^4 - \cdots$