### Further Differentiation

**AH Maths Exam Questions**

<table>
<thead>
<tr>
<th>Source: 2019 Specimen P2 Q2 AH Maths</th>
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<tbody>
<tr>
<td>(1)</td>
</tr>
<tr>
<td>(a) Given $f(x) = \sin^{-1}3x$, find $f'(x)$.</td>
</tr>
<tr>
<td>(b) For $y \cos x + y^2 = 6x$, use implicit differentiation to find $\frac{dy}{dx}$.</td>
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<tr>
<td><strong>Answers:</strong></td>
</tr>
<tr>
<td>(a) $f'(x) = \frac{3}{\sqrt{1 - 9x^2}}$</td>
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<table>
<thead>
<tr>
<th>Source: 2019 Specimen P2 Q4 AH Maths – Same as 2018 Q6</th>
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<tbody>
<tr>
<td>(2)</td>
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<tr>
<td>On a suitable domain, a curve is defined parametrically by $x = t^2 + 1$ and $y = \ln(3t + 2)$. Find the equation of the tangent to the curve where $t = -\frac{1}{3}$.</td>
</tr>
<tr>
<td><strong>Answers:</strong></td>
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<tr>
<td>$y = -\frac{9}{2}x + 5$</td>
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<table>
<thead>
<tr>
<th>Source: 2019 Specimen P2 Q10 AH Maths – Same as 2017 Q11</th>
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<tbody>
<tr>
<td>(3)</td>
</tr>
<tr>
<td>Given $y = x^{2x^3 + 1}$ where $x &gt; 0$, find $\frac{dy}{dx}$.</td>
</tr>
<tr>
<td>Write your answer in terms of $x$.</td>
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<tr>
<td><strong>Answer:</strong></td>
</tr>
<tr>
<td>$\frac{dy}{dx} = x^{2x^3 + 1}\left(6x^2 \ln x + \frac{2x^3 + 1}{x}\right)$</td>
</tr>
</tbody>
</table>
### Source: 2019 Q5 AH Maths

(4) For \( x = \ln(2t + 7) \) and \( y = t^2 \), \( t > 0 \), find

(a) \( \frac{dy}{dx} \)

(b) \( \frac{d^2y}{dx^2} \).

**Answers:**

(a) \( 2t^2 + 7t \)

(b) \( \frac{1}{2}(2t + 7)(4t + 7) \)

### Source: 2019 Q10 AH Maths

(5) A curve is defined implicitly by the equation \( x^2 + y^2 = xy + 12 \).

(a) Find an expression for \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).

(b) There are two points where the tangent to the curve has equation \( x = k \), \( k \in \mathbb{R} \). Find the values of \( k \).

**Answers:**

(a) \( \frac{dy}{dx} = \frac{y - 2x}{2y - x} \)

(b) \( k = \pm 4 \)

### Source: 2016 Q11 AH Maths

(6) The height of a cube is increasing at the rate of 5 cm s\(^{-1}\).

Find the rate of increase of the volume when the height of the cube is 3 cm.

**Answer:** \( \frac{dv}{dt} = 135cm^3s^{-1} \)
An engineer has designed a lifting device. The handle turns a screw which shortens the horizontal length and increases the vertical height.

The device is modelled by a rhombus, with each side 25 cm. The horizontal length is \( x \) cm, and the vertical height is \( h \) cm as shown.

(a) Show that \( h = \sqrt{2500 - x^2} \).

(b) The horizontal length decreases at a rate of 0.3 cm per second as the handle is turned.

Find the rate of change of the vertical height when \( x = 30 \).

Answers:  
(a) \( \text{Proof} \)  
(b) \( \frac{dh}{dt} = \frac{9}{40} \text{cm s}^{-1} \)
The position of a particle at time \( t \) is given by the parametric equations
\[ x = t \cos t, \quad y = t \sin t, \quad t \geq 0. \]

(a) Find an expression for the instantaneous speed of the particle.

(b) Calculate the instantaneous speed of the particle at point A.

\[
\text{Answers: } \quad (a) \quad \sqrt{1 + t^2} \quad (b) \quad \text{Speed} = \sqrt{1 + 9\pi^2}
\]

The equation \( x^4 + y^4 + 9x - 6y = 14 \) defines a curve passing through the point A(1, 2).

Obtain the equation of the tangent to the curve at A.

\[ y = -\frac{1}{2}x + \frac{5}{2} \]
<table>
<thead>
<tr>
<th>Source: 2015 Q6 AH Maths</th>
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<tbody>
<tr>
<td>(10) For ( y = 3^{x^2} ), obtain ( \frac{dy}{dx} ).</td>
</tr>
<tr>
<td>Answer: ( \frac{dy}{dx} = 2x \ln 3 \cdot 3^{x^2} ) or ( 2x \ln 3 \cdot e^{x^2 \ln 3} )</td>
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<th>Source: 2015 Q8 AH Maths</th>
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<tbody>
<tr>
<td>(11) Given ( x = \sqrt{t + 1} ) and ( y = \cot t ), ( 0 &lt; t &lt; \pi ), obtain ( \frac{dy}{dx} ) in terms of ( t ).</td>
</tr>
<tr>
<td>Answer: ( \frac{dy}{dx} = -2\sqrt{t + 1} \cdot \csc^2 t )</td>
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<th>Source: 2014 Q4 AH Maths</th>
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<tbody>
<tr>
<td>(12) Given ( x = \ln(1 + t^2) ), ( y = \ln(1 + 2t^2) ) use parametric differentiation to find ( \frac{dy}{dx} ) in terms of ( t ).</td>
</tr>
<tr>
<td>Answer: ( \frac{dy}{dx} = \frac{2(1 + t^2)}{1 + 2t^2} )</td>
</tr>
</tbody>
</table>
Source: 2014 Q6 AH Maths

| (13) | Given \( e^y = x^3 \cos^2 x \), \( x > 0 \), show that
\[
\frac{dy}{dx} = \frac{a}{x} + b \tan x,
\]
for some constants \( a \) and \( b \).

State the values of \( a \) and \( b \).

Answers: \( a = 3 \), \( b = -2 \)

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Source: 2013 Q11 AH Maths

| (14) | A curve has equation
\[
x^2 + 4xy + y^2 + 11 = 0.
\]

Find the values of \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) at the point \((-2, 3)\).

Answer:
\[
\frac{dy}{dx} = 4, \quad \frac{d^2 y}{dx^2} = 33
\]
### Source: 2012 Q12 AH Maths

**(15)**

The radius of a cylindrical column of liquid is decreasing at the rate of 0.02 m s\(^{-1}\), while the height is increasing at the rate of 0.01 m s\(^{-1}\).

Find the rate of change of the volume when the radius is 0.6 metres and the height is 2 metres.

*Recall that the volume of a cylinder is given by \(V = \pi r^2 h\).*

**Answer:** 

\[
\text{Rate of change of the volume} = -0.0444\pi \text{ m}^3\text{s}^{-1}
\]

### Source: 2010 Q13 AH Maths

**(16)**

Given \(y = t^3 - \frac{5}{2}t^2\) and \(x = \sqrt{t}\) for \(t > 0\), use parametric differentiation to express \(\frac{dy}{dx}\) in terms of \(t\) in simplified form.

Show that \(\frac{d^2y}{dx^2} = at^2 + bt\), determining the values of the constants \(a\) and \(b\).

Obtain an equation for the tangent to the curve which passes through the point of inflexion.

**Answers:**

\[
\frac{dy}{dt} = 6t^2 - 10t^{\frac{3}{2}}
\]

\(a = 30, \quad b = -30\)

\(2y + 8x = 5\)
(17) The curve \( y = x^{2x^2 + 1} \) is defined for \( x > 0 \). Obtain the values of \( y \) and \( \frac{dy}{dx} \) at the point where \( x = 1 \).

Answer: \( \frac{dy}{dx} = 3 \)

(18) (a) Differentiate \( f(x) = \cos^{-1} (3x) \) where \( -\frac{1}{3} < x < \frac{1}{3} \).

(b) Given \( x = 2 \sec \theta, y = 3 \sin \theta \), use parametric differentiation to find \( \frac{dy}{dx} \) in terms of \( \theta \).

Answers:

\[
(a) \quad f'(x) = \frac{-3}{\sqrt{1 - 9x^2}} \\
(b) \quad \frac{dy}{dx} = \frac{3\cos^3 \theta}{2\sin \theta}
\]

(19) A curve is defined by the equation \( xy^2 + 3x^2y = 4 \) for \( x > 0 \) and \( y > 0 \). Use implicit differentiation to find \( \frac{dy}{dx} \). Hence find an equation of the tangent to the curve where \( x = 1 \).

Answers:

\[
\frac{dy}{dx} = \frac{-y^2 - 6xy}{2xy + 3x^2}
\]

\[5y + 7x = 12\]
A curve is defined by the parametric equations \( x = \cos 2t, \ y = \sin 2t, \ 0 < t < \frac{\pi}{2} \).

(a) Use parametric differentiation to find \( \frac{dy}{dx} \).

Hence find the equation of the tangent when \( t = \frac{\pi}{8} \).

(b) Obtain an expression for \( \frac{d^2y}{dx^2} \) and hence show that \( \sin 2t \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 = k \), where \( k \) is an integer. State the value of \( k \).

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### Answers:

(a) \[ \frac{dy}{dx} = \frac{2 \cos 2t}{-2 \sin 2t} = -\cot 2t \]

**Equation of tangent:** \( x + y = \sqrt{2} \)

(b) \[ \frac{d^2y}{dx^2} = \frac{-1}{\sin^3 2t}, \ k = -1 \]